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FURTHER REMARKS ON VACCINATION FOR NON FATAL SUSCEPTIBLE INVECTIVE SUSCEPTIBLE (SIS) DISEASE

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ABSTRACT

We modify existing SIS model to reflect permanent immunity due to vaccination or previous non-fatal attacks of the disease. The model incorporates information on current and past states of disease needed by families to deciding on whether to vaccinate or not their children. Of particular interest in the analysis are the criteria for stable endemic equilibrium.

KEYWORDS: Permanent Immunity, Current and Past Information

INTRODUCTION

It is believed that the more we know how disease spread, the more we will be able to control the spread. We know that vaccines have helped to curtail the spread of diseases and typical example is the eradication of smallpox [2]. It is known that availability of vaccines may not be sufficient for people to take the necessary advantage. So in some cases, families have to be motivated to take advantage of the vaccines. The history of many vaccinations show that the progress toward increasing degrees of disease control is inter-mixed by episodes of coverage decrease due to the tension between public health targets and individual freedom i.e between compulsory vaccination and conscientious or philosophical exception [2,4]. Buonomo et al [2] further pointed out that societies are increasingly facing more complex challenge of rational exception, consisting in the family decision not to vaccinate children as a consequences pseudo- rational comparison between the risk of infection and the risk of vaccine related side effects. What is clear is that under a voluntary vaccination policy, rational exception makes eradication impossible and may trigger childhood disease to oscillate. Because of important of vaccines in the eradication of diseases and the possibility of side effects of the vaccination, the search for better knowledge of how vaccines work continues and existing mathematical models are modified in the process.

In this paper we investigate an SIS model with information–dependent function that measures how information on current and past states of the disease is used by families in deciding on whether to vaccinate or not their children.

Mathematical Model

As in [2], we envisage a full effective vaccine providing a long life immunity:

$$S' = \mu(1-p_0) - \frac{\beta(t)SI}{\phi(I)} - \mu S + \gamma I \tag{1}$$

$$I' = \frac{\beta(t)SI}{\phi(I)} - (\mu - \gamma)I \tag{2}$$

Where

S: Susceptible Individuals

I: Infective Individuals

μ: Recruitment Rate of Individuals (Newborns and Immigrants).It is also the Death Rate

P_o: Fraction of Individuals that are Vaccinated

β(t): Rate at Which Susceptible Individuals Become Infected

γ: Rate at Which Infected Individuals are Treated on Recovery

ϕ(I): Summarises how Information on Current and Past States of the Disease is Used by the Families in Deciding on Whether to Vaccinate or not their Children

We assume that the continuous function ϕ(I) satisfies

ϕ(I)=$1+\phi_1(I)$, $\phi_1(I)$ is a function of I, $\phi_1(0) = 0$, $\phi_1'(I) \geq 0$

MATHEMATICAL ANALYSIS

The population N(t) is constant. So if V(t) is the fraction of individuals that have been vaccinated, then $V(t)+S(t)+I(t)=1$ and $S+I \leq 1-p_o$

Following [2], we let

$$\bar{\beta} = \beta, \textit{in case of constant rate, and}$$
$$\textit{otherwise}$$
$$\bar{\beta} = \frac{1}{\theta}\int_0^\theta \beta(u)du$$
$$\textit{where } \theta \textit{ is the periodicity of the rate of tansmission } \beta(t)$$

Disease Free Equilibrium (DFE) and its Properties

DFE=$(1-p_o,0)$

$Let\ R_l = \frac{(1-p_o)\bar{\beta}}{\mu+\gamma}$, $R_o = \frac{\beta}{\mu+\gamma}$ is the basic reproduction number.

So, it suffices to take $\beta(t)$ as a constant.

Proposition 1

- Stability of disease free does not depend on $\phi(I)$
- If $R_l < 1$ then DFE is globally asymptotically stable

- IF $R_l \geq 1$ then DFE is unstable

Proof

DFE =(1-p_o,0)=(S_o,0)

Let x= S-S_o, then

$$x' = -\mu x - \bar{\beta} S_o I + \gamma I + \text{non linear terms}$$
$$I' = \bar{\beta} S_o I - (\mu + \gamma) I + \text{non linear terms}$$

The non linear terms tend to zero as I→0, x→0

So the relevant matrix is

$$A = \begin{pmatrix} -\mu & -\bar{\beta} S_0 + \gamma \\ 0 & \bar{\beta} S_0 - \mu - \gamma \end{pmatrix}$$

Hence stability of DFE does not depend on $\phi(t)$ proving (i)

Also $|A - \lambda I| = 0$ *gives*

$$\lambda_1 = -\mu, \lambda_2 = \bar{\beta} S_0 - \mu - \gamma \text{ and } S_o = 1 - p_o$$

So the DFE is globally asymptotically stable if $R_l < 1$ and unstable if $R_l \geq 1$.

Endemic Equilibrium

We assume that $\phi(I) = 1 + a_1 I + a_2 I^2 + \ldots + a_n I^n$ (3)

is a polynomial of degree n, the coefficients $a_1, a_2, a_3 \ldots a_n$ are non negative

Preposition 2: If $R_l \geq 1$, there is a unique endemic equilibrium EE=(S_e, I_e),

Where $S_e = R_0^{-1} \phi(I_e)$ *and* I_e *is a unique* positive solution of the equation

$$\mu(1 - p_o) - \mu I - \mu R_o^{-1} \phi(I) = 0 \quad (4)$$

Proof: Clearly, if I'=0, then equation (2) gives $S_e = \frac{(\mu + \gamma)}{\beta} \phi(I_e) = R_0^{-1} \phi(I_e)$. Substituting S into (1), we obtain

$$\mu(1 - p_o) - \mu I - \mu R_o^{-1} \phi(I) = 0 \quad (5)$$

Equation (5) implies: $a_n I^n + a_{n-1} I^{n-1} + \ldots + a_2 I^2 + R_o I - R_o(1 - p_o) = 0$ (6)

Since ai's i=1,2,3...n are non negative then by Descartes' rule of signs (6) has a unique positive root I_e.

Theorem 1: Thye endemic positive equilibrium (S_e,I_e) is globally asymptotically stable if $\frac{\phi'(I)}{\phi(I)} > \frac{\mu}{\mu+\gamma}$ (7)

Proof: Let x=S-S_e, y=I-I_e

Then

$$\frac{dx}{dt} = \frac{\beta I_e x}{\phi(I_e)} - \mu x - \frac{\beta S_e y}{\phi(I_e)} + \gamma y + \frac{\beta S_e I_e}{\left(\phi(I_e)\right)^2} y\left(\phi'(I_e)\right) + non\, linear\, term$$

$$\frac{dy}{dt} = \frac{\beta I_e x}{\phi(I_e)} + \frac{\beta S_e y}{\phi(I_e)} - (\mu+\gamma) - \frac{\beta S_e I_e}{\left(\phi(I_e)\right)^2} y\left(\phi'(I_e)\right) + non\, linear\, term$$

The linear part is $$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{\beta I_e}{\phi(I_e)} - \mu & -\frac{\beta S_e}{\phi(I_e)} + \gamma + \frac{\beta S_e I_e}{\left(\phi(I_e)\right)^2}\left(\phi'(I_e)\right) \\ \frac{\beta I_e}{\phi(I_e)} & \frac{\beta S_e}{\phi(I_e)} - (\mu+\gamma) - \frac{\beta S_e I_e}{\left(\phi(I_e)\right)^2}\left(\phi'(I_e)\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This give a characteristic roots $\lambda = \frac{(\alpha + I' + 2\mu) \pm \sqrt{\alpha I + I'^2 + \mu^2}}{2}$ (8)

Clearly $\alpha > 0, \mu > 0\, and\; I' > 0\; if\; \frac{\phi'(\mathrm{I})}{\phi(\mathrm{I})} > \frac{\mu}{\mu+\gamma}$

That is the two λ's are negative

Hence the EE is GAS

This completes the proof.

Appropriate Lyapunov Function for Endemic Equilibrium

Let $\phi(I) = e^{\alpha t}\;, \alpha > 0$

Assume that the endemic positive equilibriumis$(\mathrm{S_e}, \mathrm{I_e})$
Then if $\mathrm{x = S - S_e}, \mathrm{y = I - I_e}$

Then

$$\frac{dx}{dt} = -\beta I_e e^{-\alpha I_e - ay} - \beta S_e e^{-\alpha I_e - ay} - \mu x + \gamma y + non\ linear\ term \tag{9}$$

$$\frac{dy}{dt} = \beta I_e e^{-\alpha I_e - ay} + \beta S_e y e^{-\alpha I_e - ay} - (\mu+\gamma) y + non\ linear\ term \tag{10}$$

Theorem 2

Let $V(\mathrm{x},\mathrm{y}) = \mathrm{x}^2 + \mathrm{axy} + \mathrm{by}^2$ (11)

$$such\ that\ b - \frac{a^2}{4} > 0 \tag{12}$$

Then V(x,y) is a Lyapunov function for the system (9) and (10) if

$$a(1+I_e) < 2I_e, b < 1, I_e = S, a\gamma - 2b(\gamma+\mu) < 0 \text{ and } 2\gamma - \mathrm{a}(\mu+\gamma) = 0 \tag{13}$$

Proof

$$\begin{aligned}\frac{dV}{dt} &= xye^{-\alpha I_e - \alpha y}\left[2\beta I_e + a\beta I_e - 2\beta S_e - a\beta I_e\right] + \left[a\beta I_e + a\beta - 2\beta I_e\right]x^2 e^{-\alpha I_e - \alpha y} \\ &+ \left[2\beta b S_e + a\beta S_e\right]y^2 e^{-\alpha I_e - \alpha y} - 2\mu x^2 + \left(a\gamma - 2\mu b - 2\gamma b\right)y^2 \\ &+ \left(2\gamma - \mu_a - a\gamma\right)xy\end{aligned}$$

$$Thus\ \frac{dy}{dx} < 0\ if\ \ (13)\ holds$$

This completes the proof.

Theorem 3

(S_e, I_e) is asymptotically stable if (13) holds

Proof

This follows from the fact that V(x,y) is a Lyapnunov function.

DISCUSSIONS OF RESULTS

When the information no current and past states of the disease is sufficient then ($\alpha \to \infty$ and $\phi \to \infty$) the endemic equilibrium is unconditionally stable. When $\alpha=0$, no information on the history of the disease, then the endemic equilibrium is not automatically stable. In fact the stability of the endemic equilibrium depends on the value of the equilibrium point, the recovery rate γ and the recruitment rate μ of the new born. This study shows that information on past and present states of the disease is vital in the eradication of the disease.

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